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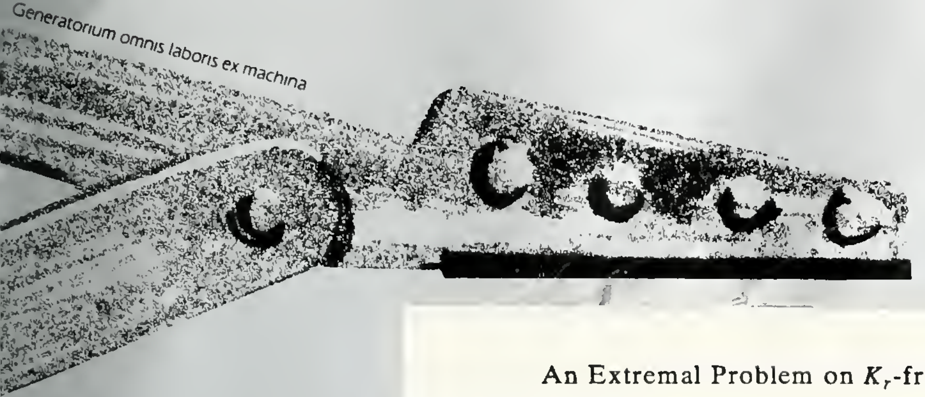

An Extremal Problem on K_r-free Graphs

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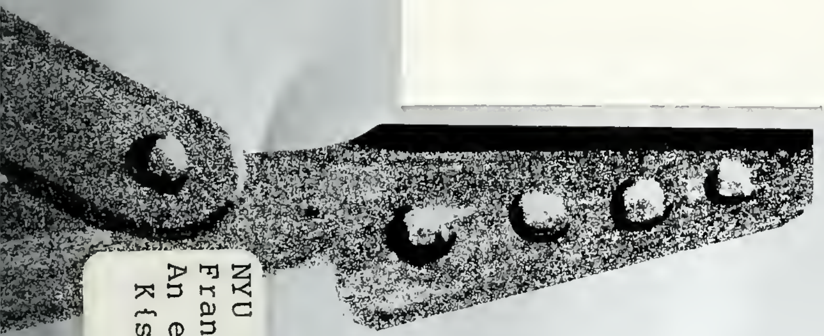

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New York University
C t Institute of Mathematical Sciences

Computer Science Division
251 Mercer Street New York, N.Y. 10012

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An extremal problem on K_r-free graphs

Peter Frankl and János Pach

CNRS, Paris and Courant Institute, New York

1. Introduction

At a meeting in Fertőd (Hungary, July 1986) L. Babai, M. Simonovits and J. Spencer [1] proved the following conjecture of P. Erdős [2].

Theorem 1 Given a random graph G_n, whose edges are chosen independently of each other with probability 1/2, let $h_0(G_n)$ and $h_1(G_n)$ be defined as the maximum number of edges of a triangle-free subgraph of G_n and of a bipartite subgraph of G_n, respectively. Then $h_0(G_n) = h_1(G_n)$ holds with probability tending to 1, as $n \rightarrow +\infty$.

It would be desirable to obtain a similar result, which is valid not only for *almost every* graph but for *all* graphs satisfying certain natural conditions. As a first step in this direction, they proposed the following problem.

Given a natural number t and $c>0$, a graph G with n vertices is said to have *property* $P(t,cn)$, if any t vertices of G have at most cn common neighbors. It is plain that e.g. a random graph G_n satisfies property $P(t,cn)$ with probability tending to 1, provided $c>2^{-t}$.

Erdős, Babai, Simonovits and Spencer conjectured that the following result is true (for $r = 3$).

Theorem 2 Let $t,r \geq 2$ be fixed natural numbers, $0<c\leq 1-\frac{1}{r-1}$. Then every K_r-free graph G with n vertices and with property $P(t,cn)$ has at most

$$\frac{c^{\frac{1}{t}}}{2}\left(1-\frac{1}{r-1}\right)^{1-\frac{1}{t}} n^2+o(n^2)$$

edges, and this bound is asymptotically sharp.

Note that the complete (r-1)-partite graph $K_{r-1}(n)$, whose classes are as equal as possible, satisfies property $P(t,cn)$ for every $c>1-\frac{1}{r-1}$ and, by Turán's theorem, its number of edges is maximal among all K_r-free graphs with n vertices.

To show that the bound in Theorem 2 is asymptotically sharp if $c\le 1-\frac{1}{r-1}$, take $K_{r-1}(n)$ and remove its edges independently of each other with probability $1-c^{\frac{1}{t}}\frac{(r-1)}{(r-2)}^{\frac{1}{t}}$. It is easy to see that the resulting graph has property $P(t,cn+o(n))$ and its number of edges is $\frac{c^{\frac{1}{t}}}{2}(1-\frac{1}{r-1})^{1-\frac{1}{t}} n^2+o(n^2)$, with probability tending to 1, as $n\to+\infty$

2. Proof of Theorem 2

The main tool of the proof is Szemerédi's Regularity Lemma [4]. Given a graph G and two disjoint subsets V_1, $V_2\subseteq V(G)$, let $e(V_1,V_2)$ denote the number of edges with one endpoint in V_1 and the other in V_2. The *edge-density* between V_1 and V_2 is

$$\delta(V_1,V_2)=\frac{e(V_1,V_2)}{|V_1|\cdot|V_2|}$$

The pair (V_1,V_2) is called ϵ-regular if

$$|\delta(V_1',V_2')-\delta(V_1,V_2)|<\epsilon$$

for every $V_1'\subseteq V_1$ and $V_2'\subseteq V_2$ such that $|V_1'|\ge\epsilon|V_1|$ and $|V_2'|\ge\epsilon|V_2|$. V_1 and V_2 are said to be *almost equal* if $||V_1|-|V_2||\le 1$.

Szemerédi's Regularity Lemma Given any $\epsilon>0$ and any natural number m, there exist $M = M(\epsilon,m)$ and $n_0 = n_0(\epsilon,m)$ with the property that the vertex set of every graph G with

$|V(G)|>n_0$ can be partitioned into almost equal classes $V_1,V_2,\ldots,V_k$ such that $m\le k\le M$ and all but at most ϵk^2 pairs (V_i,V_j) are ϵ-regular.

In the rest of this section let $r,t\ge 2$, $0<\epsilon<r^{-r}$ and $m>\epsilon^{\frac{-1}{r}}$ be fixed, and let G be a K_r-free graph with

$$n> \max\left(n_0(\epsilon,\, m),\ \frac{M(\epsilon,m)t^2}{\epsilon}\right) \tag{1}$$

vertices and having property $P(t,cn)$ for some $0<c<\dfrac{r-2}{r-1}$.

Let us consider a partition $V(G) = V_1\bigcup V_2\bigcup\cdots\bigcup V_k$ meeting the requirements of the Regularity Lemma, and define a graph G^* on the classes of this partition by

$$\begin{gathered} V(G^*) = \{V_1,V_2,\ldots,V_k\}\ \ \delta(V_i,V_j) > \epsilon^{\frac{1}{r}}+\epsilon\}\text{ and} \\ V_iV_j\in E(G^*) \Leftrightarrow \{(V_i,V_j)\ \textit{is an } \epsilon-\textit{regular pair} \end{gathered} \tag{2}$$

2.2 Proposition G^* is K_r-free.

Proof Assume, in order to obtain a contradiction, that $V_1,V_2,\ldots,V_r$, say, induce a complete subgraph of G^*. We are going to prove by induction on $j\le r$ that, given any j subsets $V_i'\subseteq V_i$, $|V_i'|\ge\epsilon^{1-\frac{j}{r}}|V_i|$, $1\le i\le j$, one can pick $v_i\in V_i'$ such that they form a K_j in G. For $j=r$ this will yield the desired contradiction.

For $j=1$ the assertion is trivial. Suppose we have already proved it for some $j<r$, and fix $j+1$ subsets $V_i'\subseteq V_i$, $|V_i'| = \left\lceil\epsilon^{1-\frac{j+1}{r}}|V_i|\right\rceil$, $1\le i\le j+1$. It is easily seen that there exists a point $v_{j+1}\in V_{j+1}'$ which has at least $\epsilon^{1-\frac{j}{r}}|V_i|$ neighbors in each V_i' $(1\le i\le j)$. Otherwise, we could pick an i such that at least $\dfrac{|V_{j+1}'|}{j}\ge\epsilon\,|V_{j+1}|$ vertices of V_{j+1}' have fewer than $\epsilon^{1-\frac{j}{r}}|V_i|$ neighbors in V_i'. Denoting the set of these vertices by V_{j+1}^0, this would imply that

$$\delta(V_{j+1}^0, V_i') = \frac{e(V_{j+1}^0, V_i')}{|V_{j+1}^0|\,|V_i'|} \leq \epsilon^{\frac{1}{r}},$$

contradicting our assumption that (V_{j+1}, V_i) is an ϵ-regular pair with $\delta(V_{j+1}, V_i) > \epsilon^{\frac{1}{r}} + \epsilon$.

Let V_i'' denote the set of all points in V_i' which are connected to v_{j+1} $(1 \leq i \leq j)$. Applying the induction hypothesis to them we conclude that the assertion remains true for $j+1$. □

2.3 Proposition If G has property $P(t, cn)$, then for every i $(1 \leq i \leq k)$

$$\sum_{j: V_iV_j \in E(G^*)} (\delta(V_i, V_j))^t \leq kc(1 + 2\epsilon^{\frac{1}{r}})$$

Proof Given any unordered t-tuple $\{x_1, x_2, \ldots, x_t\} \subseteq V_i$, let $T(\{x_1, x_2, \ldots, x_t\})$ be the number of vertices joined to all x_s $(1 \leq s \leq t)$. For every $y \in V(G) - V_i$, let $d_i(y)$ be the number of neighbors of y in V_i. Assume further, for simplicity, that k is a divisor of n. Then

$$\sum_{\{x_1, \ldots, x_t\}} T(\{x_1, \ldots, x_t\}) \geq \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \sum_{y \in V_j} \binom{d_i(y)}{t}$$

$$\geq \sum_{\substack{1 \leq j \leq k \\ j \neq i}} |V_j| \binom{\sum d_i(y)/|V_j|}{t} = \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \frac{n}{k} \binom{\delta(V_i, V_j)n/k}{t}$$

$$\geq \binom{n/k}{t} \sum_{j: V_iV_j \in E(G^*)} \frac{n}{k} \left(\delta(V_i, V_j) - \frac{kt}{n}\right)^t \geq \binom{n/k}{t} \sum_{j: V_iV_j \in E(G^*)} \frac{n}{k} (\delta(V_i, V_j))^t \, (1 - \epsilon^{1 - \frac{1}{r}}),$$

where the last inequality follows by (1) and (2). Thus there exists a specific t-tuple $\{x_1, x_2, \ldots, x_t\} \subseteq V_i$ such that

$$\sum_{j: V_iV_j \in E(G^*)} \frac{n}{k} (\delta(V_i, V_j))^t \, (1 - \epsilon^{1 - \frac{1}{r}}) \leq T(\{x_1, \ldots, x_t\}) \leq cn$$

whence 2.3 follows immediately. □

Since any edge of G connects (a) either two classes (V_i, V_j) such that $V_iV_j \in E(G^*)$, (b) or an ϵ-irregular pair (V_i, V_j), (c) or a pair (V_i, V_j) with $\delta(V_i, V_j) \leq \epsilon^{\frac{1}{r}} + \epsilon$, (d) or two points belonging to the same class V_i, we obtain

$$|E(G)| \leq \frac{n^2}{k^2}\left(\sum_{V_iV_j \in E(G^*)} \delta(V_i,V_j) + \epsilon k^2 + (\epsilon^{\frac{1}{r}} + \epsilon)\binom{k}{2} + \frac{k}{2}\right) \tag{3}$$

$$\leq \frac{n^2}{k^2} \sum_{V_iV_j \in E(G^*)} \delta(V_i,V_j) + 3\epsilon^{\frac{1}{r}} n^2.$$

By proposition 2.2, G^* is K_r-free, therefore applying Turán's theorem [5]

$$|E(G^*)| \leq \frac{r-2}{2(r-1)}\, k^2. \tag{4}$$

Hence, by (3) and (4)

$$\sum_{V_iV_j \in E(G^*)} (\delta(V_i,V_j))^t \geq |E(G^*)| \left(\frac{\sum \delta(V_i,V_j)}{|E(G^*)|}\right)^t. \tag{5}$$

$$\geq \frac{1}{|E(G^*)|^{t-1}} \left[(|E(G)| - 3\epsilon^{\frac{1}{r}} n^2)\, \frac{k^2}{n^2}\right]^t \geq \left(\frac{2(r-1)}{r-2}\right)^{t-1} \frac{k^2}{n^{2t}} (|E(G)| - 3\epsilon^{\frac{1}{r}} n^2)^t$$

On the other hand, summing up the inequalities of Proposition 2.3 for all i,

$$\sum_{V_iV_j \in E(G^*)} (\delta(V_i,V_j))^t \leq \frac{k^2}{2} c\,(1 + 2\epsilon^{\frac{1}{r}}) \tag{6}$$

Comparing (5) and (6), we obtain

$$|E(G)| \leq \left(\frac{c^{\frac{1}{t}}}{2}\Big(1 - \frac{1}{r-1}\Big)^{1-\frac{1}{t}} + 3\epsilon^{\frac{1}{r}}\right) n^2,$$

which establishes the theorem.

3. Concluding Remarks

(1) The following trivial assertion can be viewed as the special case of our Theorem 2, when $r = +\infty$. Every graph G with n vertices and with property $P(t,cn)$ has at most $c^{1/t}\binom{n}{2} + o(n^2)$ edges.

(2) We conjecture that the right order of magnitude of the remainder term in Theorem 2 is $O(n)$.

(3) Rödl [3] uses some arguments very similar to our Proposition 2.2.

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